Endogenous Cycles in the Sraffian Supermultiplier Model with a Non-linear Investment Reaction Function

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Abstract

In this paper, we examine the presence of limit cycles in a modified version of the Sraffian supermultiplier model (SSM), with a non-linear specification for the investment reaction function. We use the sensitivity of the investment share to the discrepancies between actual and normal utilization as the bifurcation parameter and identify the conditions necessary for the emergence of cycles. To accomplish this, we deliver a formal analysis of the dynamic stability of the SSM equilibrium path and present the conditions required for a Hopf bifurcation to occur. Besides, we show that the version with a non-linear investment reaction function display dynamics similar to the baseline SSM [see Araujo and Moreira (2023)].

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1. Introduction

Since Goodwin's (1967) seminal contribution, burgeoning literature highlighting the cyclical patterns as a prominent stylized fact in advanced capitalist economies arose [See e.g., Skott (1989) and Setterfield (2023)]. Self-sustained oscillations stemming from a system of non-linear differential equations are meaningful in economic terms insofar as they can replicate business cycles even in the absence of an external driving force. Such oscillations allow us to consider growth and fluctuations intertwined phenomena as in the profit squeeze mechanism, with the relevant focus placed on the intrinsic interactions among the relevant economic variables.

With the advent of techniques to detect cycles in non-linear systems, the study of oscillatory behaviour now extends beyond the elegant adaptation of the Lotka-Volterra system by Goodwin. For instance, Samuelson (1988) employed the Poincaré-Bendixson theorem to demonstrate the existence of periodic orbits in his two-dimensional multiplier-accelerator model. Skott (1989) also used this tool to prove the existence of a limit cycle in the neo-Harrodian model. Although the Poincaré-Bendixson theorem[1] provides a complete characterization of periodic orbits in two-dimensional systems (Wiggins, 2003), it has recently been supplanted by the Hopf theorem (Guckenheimer and Homes, 1983), which applies to higher dimensions as well (Araujo and Moreira, 2021; Sasaki, 2013).

Hopf bifurcation occurs when a dynamical system changes its stability as a bifurcation parameter is varied through a critical value, leading to the emergence of a periodic orbit, whose existence depends on the non-linearities and the values of the system's parameters. Accordingly, a possible kind of Hopf bifurcation occurs when a stable fixed point in the system becomes unstable, and a stable limit cycle emerges in its place, meaning that it occurs in the vicinity of an equilibrium point of a system. This means that any economic model with damped oscillations converging to a steady-state path can give rise to permanent oscillations. While any of the model's parameters can theoretically be a candidate, it is important to choose one with empirical relevance.

Once the parameter is selected, we can assert that in the presence of a Hopf bifurcation for some interval of parameter values close to the bifurcation, closed orbits of

[1] The theorem states that if a non-linear dynamical system's trajectory is confined to a bounded region of the phase space and does not converge to a fixed point or a periodic orbit, then there must be at least one limit cycle in the phase space.

the dynamical system exist, with the oscillations being an intrinsic characteristic of the dynamic system. Indeed, if the conditions for a Hopf bifurcation are met, the probability of a stable equilibrium point being turned into a limit cycle can be high. However, the emergence of such oscillations depends heavily on initial conditions and the correct range for variation of the Hopf parameter[2]. Therefore, local analysis, along with a Hopf bifurcation criterium, serves as useful information about the economy's potential for permanent oscillatory behaviour.

From this perspective, in the present note, we study the existence of limit cycles in the Sraffian Supermultiplier Model – SSM hereafter – developed initially by Serrano[3] (1995) and Bortis (1997). The SSM regained prominence after Kaleckain authors such as Lavoie (2015) and Allain (2014) borrowed the concept of autonomous consumption to deal with Harrodian instability, ensuring the convergence of the rate of capacity utilization to its normal level in the long run. Those issues related to Kakeckian investment functions were raised by Skott (2010, 2012), who also has shown that the equilibrium solution for the degree of capacity utilization with conventional Kaleckian functions overreacts to changes in the propensity to save out of profits. This would be a square consequence of the specification of investment function in the Neo-Kaleckian growth models in which investment response to changes in the degree of capacity utilization is low and constant over time.

Notwithstanding the SSM's role in supporting the Kaleckian models to deal with some of their criticisms, it did not prevent it from being subjected to its own stress test. Authors such as Skott (2017) and Nikiforos (2018) have raised concerns about the endogeneity of capacity utilization, stock and flow consistency, and the autonomous nature of aggregate demand components in the long run. While proponents of the SSM

[2] While the existence of limit cycles depends on the system's equations and parameters, the initial conditions can also play a role in determining whether or not it occurs. For example, if the initial conditions are chosen to be near a stable fixed point of the system, the trajectory may stay close to that point and not exhibit a limit cycle. On the other hand, if the initial conditions are chosen to be sufficiently far from any stable fixed point, the trajectory may exhibit a limit cycle.

[3] The roots of this approach can be found in Hicks (1950) and were unfolded by Serrano (1995) considering that expected autonomous demand is driven by the marginal propensity to consume and the accelerator raised by the investment and the autonomous consumption.

have worked to address these issues (Freitas and Christianes, 2020; Serrano et al., 2022; Summa et al., 2023), which now is an integral part of an ongoing research project, Nikiforos et al. (2023) have recently argued that the baseline SSM cannot reproduce the business cycles that characterize capitalist economies in the long run, although recognizing the possibility of a limit cycle in the SSM with a modified investment reaction function[4].

On the empirical front, Summa et al. (2023) have challenged Nikiforos et al.'s view that investment leads the cycle, pointing out a misspecification of the investment share variable. Serrano et al. (2023, p. 20) also support the view that "(...) the model can be applied, and its performance can be evaluated not only in terms of how it explains very long-run trends but also in what regards cyclical fluctuations as well." While admitting the possibility of cycles in the SSM, its proponents argue that they are not endogenously generated by the interaction of its variables, as we consider here. Instead, in their view, cycles are the outcome of oscillations of the non-capacity creating autonomous investment.

In this vein, the present paper is to the best of our knowledge the first attempt to prove the existence of endogenous cycles in a modified version, with a non-linear specification for the investment reaction function. On the one hand, we confirm the insight by Nikiforos et al. (2023) that using a non-linear investment reaction function in terms of utilization, we can obtain a limit cycle. But, on the other hand, we show that the modified model can display dynamics like the baseline SSM. In addition to this section, the next one presents the baseline model and a modified version with a non-linear investment reactions function, focusing on the dynamic stability of the SSM equilibrium. Section 3 presents the conditions for the model to undergo a Hopf bifurcation that gives rise to permanent periodic fluctuations between its endogenous variables. Section 4 concludes.

[4] Nikiforos et al. (2023, p. 4) raised the possibility that the SSM can give rise to a limit cycle if the adjustment of the investment share is not linear but S-shaped in the utilization rate. Such a specification seems reasonable from an empirical viewpoint, and here we take this possibility into account.

2. Asymptotic Stability in the Sraffian Supermultiplier

In what follows we present a detailed presentation of the baseline SSM following Serrano and Freitas (2017) to focus on its dynamic properties. We also change the model, by introducing a non-linearity in the investment reaction function to compare the dynamics with that of the baseline model. We deal initially with a closed economy without government. The economy uses capital, K, and labour, L, to produce output, Y, by using a fixed coefficient technology:

$$Y = min\left[\frac{uK}{v}, \frac{L}{l}\right] \tag{1}$$

where u is the rate of capacity utilization, v is the capital-full capacity output ratio and l is the labour-output ratio. Disregarding depreciation[5], we can write the growth rate of the stock of capital as:

$$\widehat{K} = \left[\frac{I/Y}{v}\right] u \tag{2}$$

It is assumed that the level of aggregate real investment is a fraction h of the income, namely:

$$I = hY \tag{3}$$

where h is the marginal propensity to invest. By substituting (3) into (2) we obtain:

$$\widehat{K} = \left[\frac{h}{v}\right] u \tag{4}$$

In the baseline SSM, the marginal propensity to invest changes endogenously and linearly in response to deviations in the actual degree of capacity utilization from its normal level, denoted by μ, according to a flexible accelerator investment function, where the marginal propensity to invest changes as follows:

$$\dot{h} = h\gamma(u - \mu) \tag{5}$$

where γ is the sensitivity of the investment share to the discrepancies between actual and normal utilization, which is a parameter that measures the reaction of the growth rate of the marginal propensity to invest to the deviation of the actual degree of capacity

[5] Just to make the algebraic manipulations easier.

utilization from its normal or planned level. In what follows, we will also consider a non-linear specification for this equation, which reads as[6]:

$$\dot{h} = f(u, h) = h\left[\frac{\gamma}{1+\theta|u-\mu|}\right](u-\mu) \qquad (5)'$$

This non-linear specification reduces the speed of adjustment of h towards its steady-state value. If in the original model, this speed is given by γ, now in (5)', γ is divided by $1+\theta|u-\mu|$, meaning that changes occur according to the distance of u to the normal rate of capacity utilization. Although such a non-linear specification is plausible from an empirical point of view, the further u is from the normal rate of capacity utilisation, the lower the adjustment speed of h to its steady state.

In our opinion, non-linearities are natural components of oscillating behaviour in a dynamic system. However, we can differentiate between two types of non-linearities: those that are intrinsic to the model due to the interaction of endogenous variables and those that result from a non-linear specification for behavioural variables. If the non-linear specification is supported by theoretical or empirical evidence and has a superior fit compared to the linear one, it should be incorporated into the model. However, if the choice between linear and non-linear specifications is unclear and the benefits of the non-linear specification are not evident, we opt for the linear one, in which the dynamics of the model are only affected by the interaction of endogenous variables, and not by *ad-hoc* non-linearity assumptions on behavioural equations.

In what follows we show the dynamics of the model with such a specification are quite like the dynamics of the baseline SSM, with both admitting scope for permanent cyclical behaviour. As in Serrano and Freitas (2017), we assume that workers do not save and that capitalists save a positive fraction of their income, namely s. Besides, we assume that the consumption function is given by $C = Z + cY$, where the autonomous consumption, namely Z, is the only autonomous component of aggregate demand, and c denotes the marginal propensity to consume. In this case, the output is given by the SS according to:

[6] Of course that we could have chosen another specification carrying out the non-linearity. The choice made here is according for instance with Nikiforos et al. (2023, p. 4)

$$Y = \frac{Z}{s-h} \quad (6)$$

where $s = 1 - c$ is the marginal propensity to save. By taking logs, and differentiating equation (6), one obtains:

$$\hat{Y} = \hat{Z} + \frac{1}{s-h}\dot{h} \quad (7)$$

Using the standard notation that $\hat{Z} = g_Z$ and $\hat{Y} = g_Y$ and substituting (5) into (7), one obtains after some algebraic manipulation the growth rate of output in the baseline SSM:

$$g_Y = g_Z + \frac{1}{s-h}[h\gamma(u-\mu)] \quad (8)$$

By substituting (5)' into (7), one obtains the growth rate of output in the alternative SSM:

$$g_Y = g_Z + \frac{h\left[\frac{\gamma}{1+\theta|u-\mu|}\right](u-\mu)}{s-h} \quad (8)'$$

From the definition of capacity utilization, namely $u = \frac{Y}{Y^{fc}}$, where Y^{fc} stands for the full capacity output, one obtains after some algebraic manipulation the dynamics of the capacity utilization as:

$$\hat{u} = \hat{Y} - \hat{K} \quad (9)$$

Substituting (5) and (8) into (9) one obtains the following differential equation for the rate of capacity utilization in baseline SSM:

$$\dot{u} = u\left\{g_Z + \frac{h}{s-h}[\gamma(u-\mu)] - \left[\frac{h}{v}\right]u\right\} \quad (10)$$

Alternatively, we can substitute (5)' and (8)' into (9) to obtain the following differential equation for the rate of capacity utilization in the alternative specification for the SSM:

$$\dot{u} = g(h,u) = u\left\{g_Z + \frac{h\left[\frac{\gamma}{1+\theta|u-\mu|}\right](u-\mu)}{s-h} - \frac{h}{v}u\right\} \quad (10)'$$

Despite the changes conveyed by the non-linear investment reaction to the deviation of the degree of utilization from its normal level, systems formed by (5) and (10) and (5)' and (10)' have the same equilibrium solutions. From (5) and (5)' evaluated in steady state, one obtains the following meaningful solution for the rate of capacity utilization:

$$u^* = \mu \tag{11}$$

Equation (11) shows that utilization equals the normal rate of capacity utilization in the long run for both models. From (10) and (10)' evaluated in steady state and by considering (11), one obtains:

$$h^* = \frac{v g_Z}{\mu} \tag{12}$$

Equation (12) determines the required investment share that equalizes investment and savings for both specifications of the investment reaction function. Now we have to consider a particular Jacobian for each of the systems. For the baseline model the Jacobian is given by:

$$J(h,u) = \begin{bmatrix} \gamma(u-\mu) & \gamma h \\ u\left[\frac{\gamma s(u-\mu)}{(s-h)^2} - \frac{u}{v}\right] & g_Z + \frac{\gamma h}{(s-h)}(u-\mu) - \frac{hu}{v} + u\left[\left(\frac{\gamma h}{s-h}\right) - \frac{h}{v}\right] \end{bmatrix} \tag{13}$$

For the modified model with equations (5)' and (10)', the Jacobian is given by:

$$J(h,u) = \begin{bmatrix} \gamma\left(\frac{u-\mu}{1+\theta|u-\mu|}\right) & \gamma h\left[\frac{(1+\theta|u-\mu|)-|u-\mu|}{(1+\theta|u-\mu|)^2}\right] \\ u\left\{\left(\frac{u-\mu}{1+\theta|u-\mu|}\right)\left[\frac{\gamma s}{(s-h)^2}\right] - \frac{u}{v}\right\} & g_Z + \frac{\gamma h}{s-h}\left[\frac{(u-\mu)}{1+\theta|u-\mu|}\right] - \frac{hu}{v} + u\left[\left(\frac{\gamma h}{s-h}\right)\left(\frac{1}{(1+\theta|u-\mu|)^2}\right) - \frac{h}{v}\right] \end{bmatrix} \tag{14}$$

From (13) and (14) evaluated at the equilibrium point $(h^*, u^*) = \left(\frac{v g_Z}{\mu}, \mu\right)$, we can formulate the following proposition.

Proposition 1: The eigenvalues of (13) and (14) when the systems (5) and (10) and (5)' and (10)' respectively evaluated at the steady state solution $(h^*, u^*) = \left(\frac{v g_Z}{\mu}, \mu\right)$ are the same.

Proof. The proof is based on the fact that evaluating both (13) and (14) in the steady-state solution, namely $u^* = \mu$ and $h^* = \frac{v g_Z}{\mu}$, the Jacobian is the same for both systems and is given by:

$$J(h^*, u^*) = \begin{bmatrix} 0 & \gamma\frac{g_Z v}{\mu} \\ -\frac{\mu^2}{v} & g_Z\left(\frac{v\gamma}{s - \frac{v}{\mu} g_Z} - 1\right) \end{bmatrix} \tag{15}$$

The characteristic polynomial of the Jacobian matrix (15) at (h^*, u^*) is given by:

$$\lambda^2 - g_Z\left(\frac{\gamma\mu v-\mu s+v g_Z}{\mu s - v g_Z}\right)\lambda + \mu\gamma g_z = 0 \tag{16}$$

The roots of the characteristic polynomial are the eigenvalues of both systems, which are the same and given by:

$$\lambda_{1,2} = -\frac{1}{2}\left(\frac{\gamma v\mu}{s\mu - v g_Z} - 1\right) g_Z \pm \frac{\sqrt{(g_Z)^2\left(\frac{\gamma v\mu}{s\mu - v g_Z}-1\right)^2 - 4\mu\gamma g_Z}}{2} \tag{17}$$

Insofar as eigenvalues determine the stability of a dynamical system, systems with the same eigenvalues will behave similarly in terms of stability. ■

To study the stability of both systems, we take as starting point Gandolfo (1997, p. 254), for whom $trJ(h^*, u^*) < 0$ and $detJ(h^*, u^*) > 0$ provides sufficient conditions[7] for asymptotic stability of a two-dimensional system of differential equations, which are given respectively by:

$$trJ(h^*, u^*) = g_Z\left(\frac{v\gamma}{s - \frac{v}{\mu}g_Z} - 1\right) \tag{18}$$

$$detJ(h^*, u^*) = \mu\gamma g_Z \tag{19}$$

It is easy to see that $detJ(h^*, u^*) > 0$ due to the positivity of all parameters. To find the sign of $trJ(h^*, u^*)$, Serrano and Freitas (2017) showed that if $c + \frac{v g_Z}{\mu} + \gamma v < 1$ then $trJ(h^*, u^*) < 0$. That condition has a sound economic meaning[8] and is satisfied by assuming a sufficiently low value for the reaction parameter γ. Although the stability of the system is figured out by the trace and determinant of the Jacobian, with this information only it is not possible to fully characterise the dynamics of the system around the equilibrium point (h^*, u^*). To consider this point let us prove the following lemma.

Lemma 1. $trJ(h^*, u^*) < 0$ iff $v g_Z < \mu(s - \gamma v)$

[7] We confirm what was pointed out by Blecker and Setterfield (2019, p. 361) "stability of the Sraffian supermultiplier model thus demands that we make certain assumptions about the size of the marginal propensity to send and its components parts. These assumptions are no doubt contestable (…). Nevertheless, we can now state unequivocally that early reservations notwithstanding, there are conditions under which the Sraffian supermultiplier model is demonstrably stable."

[8] According to Serrano and Freitas (2017, p. 76): "This additional condition implies that the local dynamic stability of the fully adjusted equilibrium requires that the aggregate marginal propensity to spend in the neighbourhood of the fully adjusted equilibrium must be lower than one."

Proof. After some algebraic manipulation, we can rewrite (18) as $trJ(h^*, u^*) = \left(\frac{\gamma v \mu}{s\mu - v g_Z} - 1\right) g_Z$. Hence $trJ(h^*, u^*) < 0$ iff $\frac{\gamma v \mu}{s\mu - v g_Z} < 1$, which yields after some algebraic manipulation $v g_Z < \mu(s - \gamma v)$. ■

We can rewrite $v g_Z < \mu(s - \gamma v)$ as $\gamma < \frac{s\mu - v g_Z}{\mu v}$ to emphasise that the additional condition raised by Serrano and Freitas (2017, p. 76) to guarantee that $trJ(h^*, u^*) < 0$ implies that "the investment response of firms designed specifically to restore the capacity utilization rate to its normal level must be sufficiently weak" as pointed out by Blecker and Setterfield (2019, p. 361). It is easy to see that the conditions $trJ(h^*, u^*) < 0$ and $detJ(h^*, u^*) > 0$ implies that both roots are either negative or complex with the real part negative. The first case is obtained if the discriminant $\Delta = trJ(h^*, u^*)^2 - 4detJ(h^*, u^*) = \left(\frac{\gamma v \mu}{s\mu - v g_Z} - 1\right)^2 - 4\mu\gamma g_Z < 0$. One of the roots is negative, and the other may also be shown to be negative by considering that $trJ(h^*, u^*) + \sqrt{trJ(h^*, u^*)^2 - 4detJ(h^*, u^*)} < 0$. If it is the case, we can unambiguously say that (h^*, u^*) is a stable node insofar as the eigenvalues are real and negative.

But, we have to bear in mind that $trJ(h^*, u^*) < 0$ and $detJ(h^*, u^*) > 0$ may give rise to a second case, in which the discriminant is negative, which corresponds to the case of a stable spiral. Hence, while those conditions guarantee the asymptotical stability of (h^*, u^*), we cannot distinguish beforehand if the equilibrium is a stable node or a stable spiral. Some authors such as Nikiforos et al. (2023, p. 4) referring to the Jacobian of the baseline system consider that "[t]he oppositely signed off-diagonal entries make cycles likely: as subsequent discussion shows, eigenvalues are very likely to be complex." But their subsequence discussion is based on a heuristic analysis with a sound economic meaning but that lacks a formal demonstration. In the next proposition, we show that the necessary condition for guaranteeing that the trace is negative is the same as needed to prove that the eigenvalues are complex numbers.

Proposition 2. If $g_Z \left(\frac{\gamma v \mu}{s\mu - v g_Z} - 1\right)^2 < 4\mu\gamma$ and if $v g_Z < \mu(s - \gamma v)$ the equilibrium point $P = (h^*, u^*)$ is a *stable spiral focus.*

Proof. If $\Delta < 0$, which corresponds to the case in which the eigenvalues of $J(h^*, u^*)$ are complex conjugated roots of the characteristic polynomial. If it happens, and the real part of the complex roots is different from zero, the point $P = (h^*, u^*)$ is a hyperbolic fixed point and we can use the Hartman-Grobman linearization theorem, to prove the asymptotic stability of the system around (h^*, u^*). The real part of the complex roots is $\frac{1}{2} trJ(h^*, u^*) \neq 0$, which ensures that we can use the linearization theorem in a neighbourhood of (h^*, u^*). Besides, lemma 1 guarantees that $trJ(h^*, u^*) < 0$ if $vg_Z < \mu(s - \gamma v)$. Hence, as far as the two eigenvalues are complex with a negative real part, we can say that the equilibrium point (h^*, u^*) is a stable spiral. ■

Proposition 2 confirms, from a formal perspective, what was suggested by Nikiforos et al. (2023) – that the baseline model has an equilibrium with convergent dumped cyclical trajectories, as shown in figure 1. The modified model, equipped with equations (5)' and (10)', also exhibits the same behaviour, insofar as the eigenvalues are identical for both systems as shown in figure 2.

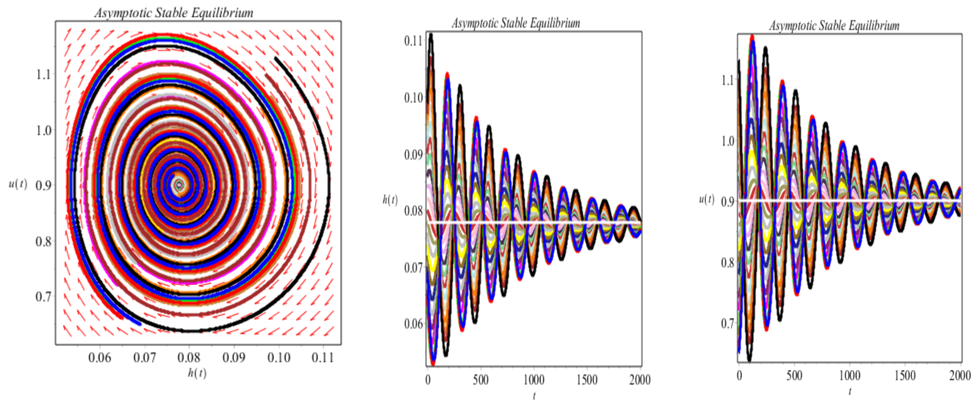


Figure 1. If $\gamma = 0.112, \mu = 0.9, g_Z = 0.02, s = 0.199, v = 3.5$, t=0...2000, the positive equilibrium $P^* = (0.07777777778, 0.9)$ is locally asymptotically stable with two eigenvalues complex $-0.00133822181250000 \pm 0.0231993353835135\, i$.

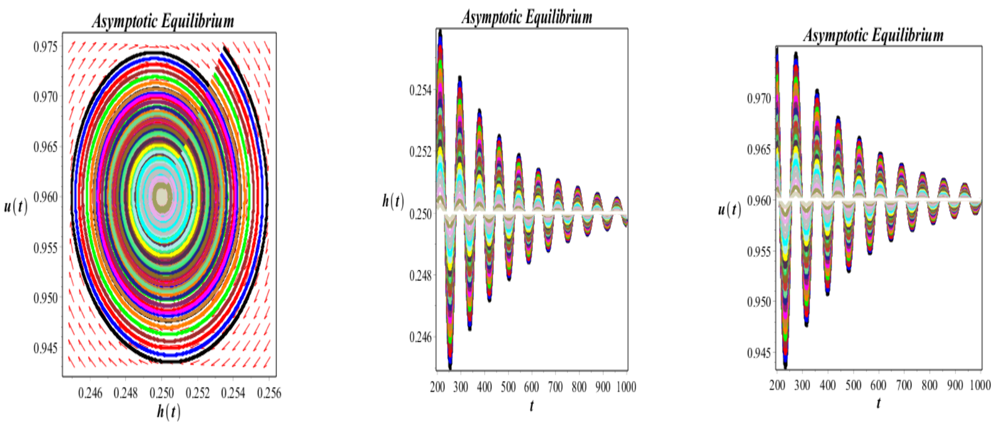


Figure 2. If $\gamma = 0.1, \mu = 0.96, g_z = 0.06, s = 0.7, \upsilon = 4, \theta = 0.4$, t=200...1000, the positive equilibrium $P^* = (0.25, 0.96)$ is locally asymptotically stable with two eigenvalues complex $-0.00333333333000000 \pm 0.0758214276369887\, i$.

3. Hopf Bifurcation in the Sraffian Supermultiplier

Bifurcation analysis involves studying the behaviour of the system as parameters are varied and identifying critical values where qualitative changes in the dynamics occur. As far as the two eigenvalues of both systems are likely to be complex, there is scope for permanent cyclical behaviour if a bifurcation occurs at that value of the parameter where the equilibrium changes from being locally stable to unstable. To prove the existence of such a limit cycle in the SSM, we can use the Hopf theorem (see, e.g, Guckenheimer and Homes, 1983), which demands that at chosen bifurcation parameter, let us say $\gamma = \gamma^*$, $\lambda_{1,2}$ become a pair of purely imaginary eigenvalues and $\frac{dRe\lambda(\gamma^*)}{d\gamma} \neq 0$. We consider that the sensitivity of the investment share to the discrepancies between actual and normal utilization, namely γ, as the most appropriate candidate for the Hopf bifurcation parameter. As shown in eq. (17), both the real and the imaginary parts of the complex eigenvalues are also a function of γ. Hence, the dynamics of the whole model are affected by this parameter and, a Hopf bifurcation asserts that for some interval of values close to its critical value, closed orbits of the dynamical system exist.

Other candidates to be the Hopf bifurcation parameter would be the normal rate of capacity utilization, namely μ, or the growth rate of autonomous demands, namely g_z. Although exogenous, there is abundant evidence pointing out to volatile behaviour of both variables. Regarding the former, the findings of Bassi et al. (2022) highlight variability in the normal rate of utilization hold over different ranges of variation in the actual rate. There are in fact two competing views concerning the endogeneity of the normal rate of capacity utilization. For authors such as Serrano (1995) and Skott (2010), it is not an endogenous variable but rather a parameter, with the actual rate adjusting towards it, as it happens in the SSM model. But for Dutt (1997) and Nikiforos (2018), it is an endogenous variable that adjusts to the actual rate. Girardi and Pariboni (2018, p. 342) challenged this view considering that "the derivation of the macroeconomic adjustment mechanism from the microeconomic analysis involves a logical leap that can be justified only by a rather peculiar aggregation process".

Insofar as our focus here is not on the controversy but rather to show the existence of permanent cycles in the SSM, we swerve into choosing this parameter as the bifurcation one. In the same vein, we could consider the possibility of using the growth rate of autonomous demands as such a parameter. Authors such as Skott (2017, 2019), and even Serrano et al. (2023) have pointed out variations in this parameter. In principle, such variations could give rise to a limit cycle if it were chosen as the bifurcation parameter. But here also to avoid getting into the controversy if such variations are at odds with the SSM, which is not central to the main point of our paper, we have chosen the sensitivity of the investment share to the discrepancies between actual and normal utilization as a viable candidate for the bifurcation parameter, and a bifurcation may occur when changes in its value occur, as outlined in the next proposition.

Proposition 3: The system of equations (5) and (10) and (5)' and (10)' undergoes a Hopf bifurcation if $vg_Z < \mu(s - \gamma v)$.

Proof. Using γ as the bifurcation parameter allows us to write both the real and the imaginary parts of the eigenvalues of the Jacobian at (h^*, u^*) as a function of the bifurcation parameter γ, namely $\lambda_{1,2} = \theta(\gamma) \pm i\omega(\gamma)$. Besides, $\theta(\gamma) = 0 \Leftrightarrow \bar{\gamma} = \frac{s\mu - vg_Z}{v\mu}$. The existence of a positive critical value to the bifurcation parameter is guaranteed if $vg_Z < \mu(s - \gamma v)$, which implies that $s\mu > vg_Z$. It makes the pair of complex conjugate eigenvalues to become purely imaginary, with no other eigenvalues with zero real part at (h^*, u^*). This means that the pair of complex conjugate eigenvalues become pure imaginary at the critical value of the parameter, with no other eigenvalues with zero real part at (h^*, u^*). Besides, it is easy to see that: $\frac{\partial\theta(\bar{\gamma})}{\partial\gamma} = \frac{1}{2}\left(\frac{\mu v g_z}{\mu s - v g_z}\right) > 0$, satisfying the transversality condition. Hence, the conditions for the existence of a Hopf bifurcation hold and the asymptotically stable equilibrium can give rise to a limit cycle as shown in figure 2. ■

To illustrate Proposition 3, as the real part of the eigenvalues is given by $\theta(\gamma) = -\frac{1}{2}\left(\frac{\gamma v \mu}{s\mu - vg_Z} - 1\right) g_Z$, which is a continuous function of γ, we can use the Intermediate Value Theorem and prove that there exists $\bar{\gamma}$ such that $\theta(\bar{\gamma}) = 0$. We can show that P^* is a hyperbolic stable focus for $\theta(\gamma_1) < 0$, considering for instance that $\gamma_1 = 0.1124999999$, which yields a stable focus $P^*(0.25, 0.96)$ with the real part of the eigenvalues being given by $\theta(\gamma_1) = -2.40000000000000\ 10^{-11} < 0$. If we choose

$\gamma_2 = 0.1125000001$, the equilibrium $P^*(0.25, 0.96)$ is an unstable focus insofar as the real part becomes positive $\theta(\gamma_2) = 2.88000000000\ 10^{-11} > 0$. The Intermediate Value Theorem applied to the function $\theta(\gamma)$ assures us that there exists at least one $\gamma = \bar{\gamma} \epsilon [\gamma_1, \gamma_2]$ such that $\theta(\bar{\gamma}) = 0$, that is, the complex eigenvalues are purely imaginary. It is possible to show that the Hopf bifurcation parameter that satisfies $\theta(\bar{\gamma}) = 0$ is given by $\gamma = \bar{\gamma} = 0.1125$. In figure 2, each colour represents particular initial conditions meaning that the existence of the cycle is robust in face of the choice made for the collection of parameters.

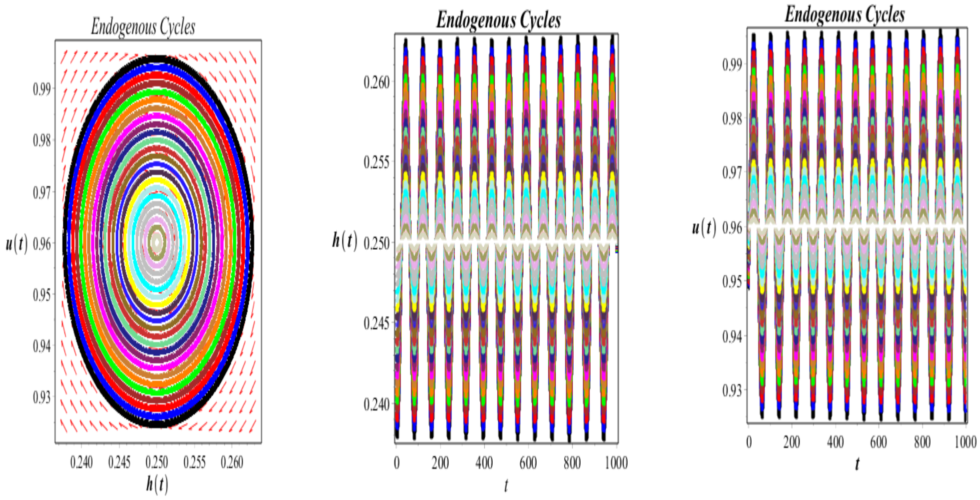


Figure 3. Endogenous cycle in the baseline model. Parameters are fixed as $\gamma = \bar{\gamma} = 0.1125, g_Z = 0.06, s = 0.7, \nu = 4, \mu = 0.96$. Here $t = 0 \dots 1000$. The unique positive equilibrium $P^* = (h^*, u^*) = (0.25, 0.96)$, with two complex conjugate and zero real part, that is $\pm 0.080498447189924\ i$

Before the bifurcation point, the stable fixed point attracts all nearby trajectories, leading to convergence towards the equilibrium. Beyond the bifurcation point, the stable periodic orbit attracts nearby trajectories, and the system exhibits periodic oscillations, which characterises a transcritical Hopf bifurcation. In the presence of such a bifurcation, the system undergoes a qualitative change in its dynamics as the bifurcation parameter crosses a critical value giving rise to a limit cycle of the dynamical system[9], whereby the oscillations are shown in figures 3 and 4 being an intrinsic characteristic of the baseline

[9] The Hopf bifurcation theorem only guarantees the existence of a small-amplitude limit cycle that arises from a supercritical Hopf bifurcation. This means that the limit cycle exists for a certain range of parameter values, and it is locally stable, but it may not be globally stable.

SSM and the modified version, namely equations (5)’ and (10)’, equipped as the non-linear investment reaction function. As the eigenvalues of this system are equal to those of the baseline model, the demonstration of the existence of a limit cycle in that model is the same, and figure 3 illustrates the cycle using different values for the parameters.

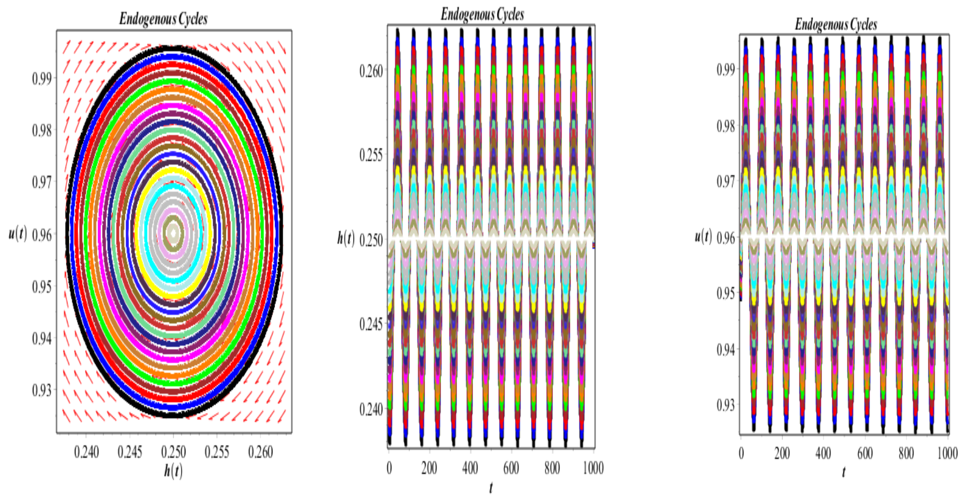


Figure 4. Endogenous cycle in the modified model. Parameters are fixed as $\gamma = \bar{\gamma} = 0.1125$, $\mu = 0.96, g_z = 0.06, s = 0.7, v = 4, \theta = 0.4,\ t = 0 \dots 1000$. The unique positive equilibrium $\mathrm{P}^* = (h^*, u^*) = (0.25, 0.96)$, has two complex conjugate eigenvalues with zero real part $\pm\, 0.0804984471899924\, i$.

Regarding the modified model, we could follow a complementary route to investigate the existence of the limit cycle – or its non-existence – by using the Dulac-Bendixson theorem. On the one hand, if the Bendixson-Dulac criterion is satisfied, it means that the planar system does not have a limit cycle in a particular region of the phase space. On the other, if the autonomous system has a region in which the divergence of the vector field is positive and a region in which the divergence is negative, and if these regions are bounded by closed curves (i.e., no trajectories can cross these curves), then it can have at least one closed trajectory or limit cycle. To use the Bendixson-Dulac criteria to prove the non-existence of a limit cycle, we need to find a Dulac function, namely $\varphi(h, u)$, which is a C^1 function such that the expression: $\frac{\partial(\varphi f)}{\partial h} + \frac{\partial(\varphi g)}{\partial u}$ has the same sign $(\neq 0)$ almost everywhere in a simply connected region of R^2 . In what follows, we show that $\varphi(h, u) = \frac{1}{hu}$ satisfies those conditions.

Proposition 4: In the modified model, equipped with the non-linear investment reaction function, there exists a Dulac function, which is a C^1 function $\varphi(x, y)$ such that the expression: $\frac{\partial(\varphi f)}{\partial h}+\frac{\partial(\varphi g)}{\partial u}$ has the same sign ($\neq 0$) almost everywhere if $\left(\frac{\gamma}{s-h}\right)\left(\frac{1}{(1+\theta|u-\mu|)^2}\right)-\frac{1}{v}\neq 0$.

Proof. Let us show that the function $\varphi(h,u)=\frac{1}{hu}$ is a Dulac function for the modified system. By multiplying $f(h,u)$ and $g(h,u)$ by it, respectively, we get the following equations:

$$\varphi f=\left[\frac{\gamma}{1+\theta|u-\mu|}\right]\frac{(u-\mu)}{u} \tag{20}$$

$$\varphi g=\left(\frac{1}{h}\right)\left(g_Z+\frac{h\left[\frac{\gamma}{1+\theta|u-\mu|}\right](u-\mu)}{s-h}-\frac{h}{v}u\right) \tag{21}$$

By taking the derivative of φf according to h, one obtains:

$$\frac{\partial}{\partial h}(\varphi f)=0 \tag{22}$$

and the derivative of φg according to u, yields:

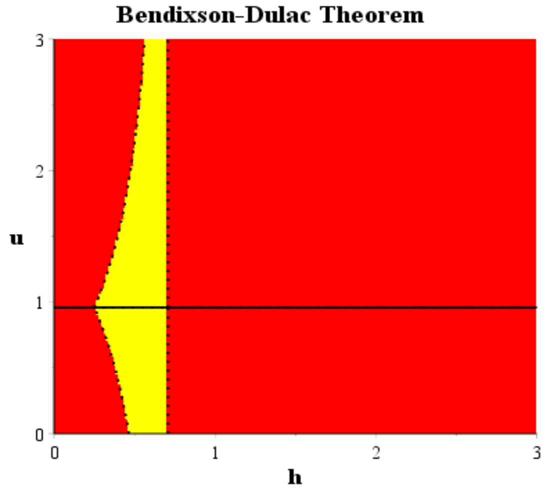
$$\frac{\partial}{\partial h}(\varphi g)=\left(\frac{\gamma}{s-h}\right)\left(\frac{1}{(1+\theta|u-\mu|)^2}\right)-\frac{1}{v} \tag{23}$$

Thus, from the hypothesis, we conclude that:

$$\frac{\partial}{\partial h}(\varphi f)+\frac{\partial}{\partial h}(\varphi g)=\left(\frac{\gamma}{s-h}\right)\left(\frac{1}{(1+\theta|u-\mu|)^2}\right)-\frac{1}{v}\neq 0 \tag{24}$$

■

If we could guarantee that $\left(\frac{\gamma}{s-h}\right)\left(\frac{1}{(1+\theta|u-\mu|)^2}\right)-\frac{1}{v}$ does not change its sign in a simply connected region of the plane, we would conclude the non-existence of a limit cycle. Using the parameters of figures 3 and 4, to illustrate the working of Proposition 4, in the red region on the right of the yellow region, we can have a simply connected set in the red region on the right of the yellow one, which allows us to conclude the non-existence of a limit cycle in that region. However, in a region close to the equilibrium point where we the modified system have a region with positive divergence (yellow), in which $\left(\frac{0.1125}{0.7-h}\right)\left(\frac{1}{(1+0.4|u-0.96|)^2}\right)-\frac{1}{4}>0$ and a region with negative divergence (red on the left of the yellow region), in which $\left(\frac{0.1125}{0.7-h}\right)\left(\frac{1}{(1+0.4|u-0.96|)^2}\right)-\frac{1}{4}<0$, does not allow us to conclude the non-existence of the limit cycle, and in fact, the existence of the cycle was proven in proposition 3 using the Hopf bifurcation theorem.



It is important to note that in the modified SSM, the growth rate of the autonomous component remains constant, giving just the trend to the output growth rate. Thus, changes in the growth rate of autonomous consumption cannot explain the cycles in the present case. Once the critical value of the bifurcation parameter is met, the system displays oscillatory behaviour, which differs from the RBC literature, where external shock effects dissipate over time. In face of that, we do not agree with the view that cyclical fluctuations do not affect 'the behaviour of the system in the long run' or that '(…) the only way to have a consistent story of the cycle with the long run theory of growth would be to resort to random shocks.' as pointed out by Nikiforos et al. (2023, p. 9). What we prove is that under fair conditions the model may undergo a Hopf bifurcation and, while such a bifurcation typically results in the emergence of a limit cycle, it is possible for a dynamical system that has undergone it to return to a stable equilibrium state[10].

4. Concluding Remarks

[10] If the bifurcation parameter is adjusted back to its initial value, it is possible for the limit cycle to disappear, and the system can return to a stable equilibrium state. In general, it is easier to return to a stable equilibrium state if the limit cycle is small and weakly attracting, rather than large and strongly attracting.

In this note, we investigated the existence of limit cycles both in a modified model with a non-linear investment reaction function. Using the sensitivity of the investment share to the discrepancies between actual and normal utilization as the bifurcation parameter, we have found that the conditions for the emergence of a bifurcation are fairly achievable from an analytical viewpoint. We are aware that concerns can be raised about the plausibility of the values of the parameters chosen, and if the critical value of the bifurcation will be met in actual economies. Not disregarding these as essential points, however, our aim here is just to show the presence of endogenous cycles in a non-linear version. Our findings do support the view that the SSM is compatible with the fact that most macroeconomic variables exhibit fluctuations around their long-term trends.

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